8.333 Fall 2025 Recitation 10: Boson and fermion gases

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This recitation serves as a review on boson and fermion gases before the final exam.

For a more comprehensive resource, see Ch. 7 of Mehran Kardar's *Statistical Physics of Particles* and Ch. 3 of David Tong's *Lectures on Statistical Physics*.

Contents

1 Boson gas

1.1 Bose-Einstein distribution

We first review the derivation of the Bose-Einstein distribution, which highlights some important concepts on canonical ensemble versus grand canonical ensemble. Consider N non-interacting bosons where each particle is in an energy eigenstate $|r\rangle$ with energy E_r. Since the particles in the same energy eigenstate are indistinguishable, we only count the number of particles n_r in state $|r\rangle$. The total energy is $\sum_r n_r E_r$. The canonical partition function then sums over all possible n_r that have the correct total number $\sum_r n_r = N$:

$$Z = \sum_{\{n_r\} | \sum_r n_r = N} e^{-\beta \sum_r n_r E_r} \,. \tag{1}$$

Here comes the potential confusion about grand canonical ensemble. Since it is hard to perform the above constrained sum explicitly, we usually "switch" to the grand canonical ensemble instead:

$$Q = \sum_{\{n_r\}} e^{-\beta \sum_r n_r (E_r - \mu)} \,. \tag{2}$$

Then, we have to impose the constraint that the average number of particles should be N, which determines μ in terms of N, T:

$$N = \frac{1}{\beta} \frac{\partial \log Q}{\partial \mu} \,. \tag{3}$$

The hope is that in the thermodynamic limit $N \to \infty$, the fluctuations in the number of particle vanish, and the grand canonical results should agree with the canonical results. However, this is merely a *mathematical* trick for studying the canonical ensemble, and the number of particles in the physical system never fluctuates. In contrast, in a genuine grand canonical ensemble, μ should be a fixed parameter instead of a function of N, T, and the number of particles do fluctuate. In this case, the average number of particles is determined by the choice of μ instead of constrained by hand.

For now, let us continue the calculation of Q. First, the partition function for a single state is

$$Q_r = \sum_{n_r=0}^{\infty} e^{-\beta n_r(E_r-\mu)} = \frac{1}{1-e^{-\beta(E_r-\mu)}} \,. \tag{4}$$

Note that for this to converge, we must assume that $E_r > \mu$. This must be satisfied for all r, so μ must be smaller than the ground state energy E_0. Now, since each r is independent, the full partition function is simply

$$Q = \prod_r \frac{1}{1-e^{-\beta(E_r-\mu)}} \,, \tag{5}$$

and the average number of particles is

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\sum_r \log\frac{1}{1-e^{-\beta(E_r-\mu)}} = \sum_r \frac{1}{e^{\beta(E_r-\mu)}-1} \,. \tag{6}$$

Note that inverting this equation gives μ as a function of N, T. Since we also have $\sum_r \langle n_r\rangle = N$, we conclude that

$$\langle n_r\rangle = \frac{1}{e^{\beta(E_r-\mu)}-1} \,. \tag{7}$$

This is the Bose-Einstein distribution.

1.2 High temperature expansion

Let us now apply the above results to the free boson gas with energy

$$E = \frac{\hbar^2 k^2}{2m} \,. \tag{8}$$

The ground state energy is $E_0 = 0$, so we require $\mu < 0$. Given that we separate out the ground state contribution, we can replace the sum of states with an integral. Denoting the fugacity $z = e^{\beta\mu}$ with $0 < z < 1$, the constraint with N now becomes

$$\begin{aligned} N &= N_{E=0} + N_{E>0} && (9)\\ &= \frac{1}{z^{-1}-1} + V\int \frac{d^3k}{(2\pi)^3}\frac{1}{z^{-1}e^{\beta\hbar^2k^2/2m}-1} && (10)\\ &= \frac{1}{z^{-1}-1} + \frac{V}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{z^{-1}e^{\beta\hbar^2k^2/2m}-1} && (11) \end{aligned}$$

Let $x = \beta\hbar^2k^2/2m$. Then,

$$N = \frac{1}{z^{-1}-1} + \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty \frac{x^{1/2}\,dx}{z^{-1}e^x-1} \,. \tag{12}$$

This equation is quite complicated, so let us study different limits of z. For $z \ll 1$, the first term is negligible in the thermodynamic limit $V \to \infty$. For the second term, smaller z is balanced by smaller β to give the same N, so this is a high temperature limit. We can expand the second term in powers of z:

$$\begin{aligned} N &\simeq \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty dx\, z x^{1/2} e^{-x}(1+ze^{-x}) && (13)\\ &= \frac{V}{\Lambda^3} z\left(1+\frac{z}{2\sqrt{2}}\right) , && (14) \end{aligned}$$

where $\Lambda = h/\sqrt{2\pi m k_B T}$ is the thermal de Broglie wavelength. The leading order solution

$$z \simeq \frac{N\Lambda^3}{V} \,, \tag{15}$$

is the same as the classical gas. The condition $z \ll 1$ then means that the inter-particle spacing is much larger than the thermal wavelength. Including the correction term, we have

$$z \simeq \frac{N\Lambda^3}{V}\left(1-\frac{1}{2\sqrt{2}}\frac{N\Lambda^3}{V}\right) . \tag{16}$$

We can now use this result to calculate the equation of state of the gas. Recall that the pressure of the gas is given by the grand free energy:

$$\begin{aligned} pV &= \frac{\log Q}{\beta} && (17)\\ &= \frac{V}{\beta}\int \frac{d^3k}{(2\pi)^3}\log\frac{1}{1-ze^{-\beta\hbar^2k^2/2m}} && (18)\\ &= \frac{V}{4\pi^2\beta}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty dx\, x^{1/2}\log\frac{1}{1-ze^{-x}} && (19)\\ &\simeq \frac{V}{4\pi^2\beta}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty dx\, zx^{1/2}e^{-x}(1+ze^{-x}/2) && (20)\\ &= \frac{V}{\Lambda^3\beta}z\left(1+\frac{z}{4\sqrt{2}}\right) && (21)\\ &\simeq Nk_BT\left(1-\frac{1}{4\sqrt{2}}\frac{N\Lambda^3}{V}\right). && (22)\end{aligned}$$

This is the same as the classical gas but with an effective attractive interaction in terms of the second virial coefficient due to the Bose statistics.

1.3 Bose-Einstein condensation

The above becomes more interesting when z is close to 1 or at low temperature, as now the first term $N_{E=0}$ may contribute significantly. Indeed, the second term satisfies

$$\begin{aligned} N_{E>0} &\leq \frac{V}{4\pi^2}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty \frac{x^{1/2}\,dx}{e^x-1} && (23)\\ &= V\left(\frac{m}{2\pi\hbar^2\beta}\right)^{3/2} f^+_{3/2}(1)\,, && (24)\end{aligned}$$

where $f^+_{3/2}(1) \simeq 2.612$. The above matches with N at the critical temperature

$$N = V\left(\frac{mk_BT_c}{2\pi\hbar^2}\right)^{3/2} f^+_{3/2}(1) \quad\Rightarrow\quad T_c = \frac{2\pi\hbar^2}{mk_B}\left(\frac{N}{f^+_{3/2}(1)V}\right)^{2/3}. \tag{25}$$

At temperature below T_c, $N_{E>0}$ becomes smaller than N, and the first term $N_{E=0}$ must contribute significantly (with z close to 1):

$$N_{E=0} = N - N_{E>0} = N\left(1-\left(\frac{T}{T_c}\right)^{3/2}\right). \tag{26}$$

This is the Bose-Einstein condensate.

To summarize, for $T > T_c$, we have $z < 1$ and the value of z is such that there are N particles in the excited states, while the number of particles at the ground state is negligible. For $T < T_c$, we have $z \simeq 1$, which determines the number of particles in the excited states; the remaining particles must condense at the ground state.

2 Fermion gas

2.1 Fermi-Dirac distribution

Consider N non-interacting fermions. Again, we will work in the grand canonical ensemble, but keep in mind that we have to impose the constraint on the average number of particles.

Due to the Pauli exclusion principle, each energy eigenstate can only be occupied by $n_r = 0, 1$ particles. Therefore, the partition function for a single state is

$$Q_r = 1 + e^{-\beta(E_r-\mu)}\,. \tag{27}$$

Note that this time we do not have any constraint on μ for convergence. The full partition function is

$$Q = \prod_r \left(1 + e^{-\beta(E_r-\mu)}\right), \tag{28}$$

and the average number of particles is

$$N = \frac{1}{\beta}\frac{\partial}{\partial\mu}\sum_r \log\left(1 + e^{-\beta(E_r-\mu)}\right) = \sum_r \frac{1}{e^{\beta(E_r-\mu)}+1}\,. \tag{29}$$

Therefore, the Fermi-Dirac distribution is

$$\langle n_r \rangle = \frac{1}{e^{\beta(E_r-\mu)}+1}\,. \tag{30}$$

2.2 Fermi surface

Let us first look at the $T = 0$ case. The Fermi-Dirac distribution reduces to

$$\langle n_r \rangle = \begin{cases} 1 & E_r < \mu \\ 0 & E_r > \mu \end{cases}\,. \tag{31}$$

That is, the particles occupy the lowest energy states without repetition. The highest energy occupied is $\mu(T = 0)$, which is also known as the Fermi energy E_F.

For free fermion gas with energy

$$E = \frac{\hbar^2 k^2}{2m}\,, \tag{32}$$

we can accordingly define the Fermi momentum k_F such that $E_F = E(k_F)$. The set of filled states satisfies $|\vec{k}| \le k_F$ and is called the Fermi sea or the Fermi sphere. Correspondingly, the Fermi surface is the set of states with $|\vec{k}| = k_F$. When there are N particles, the Fermi surface is given by

$$N = gV\int_{|\vec{k}|\le k_F} \frac{d^3k}{(2\pi)^3} = \frac{gVk_F^3}{6\pi^2} = \frac{gV}{6\pi^2}\left(\frac{2mE_F}{\hbar^2}\right)^{3/2}\,, \tag{33}$$

where g is the spin degeneracy; for example we have $g = 2$ for electrons. The total energy is

$$E = gV\int_{|\vec{k}|\le k_F} \frac{d^3k}{(2\pi)^3}\frac{\hbar^2k^2}{2m} = \frac{\hbar^2}{2m}\frac{gVk_F^5}{10\pi^2} = \frac{3}{5}NE_F\,. \tag{34}$$

That is, the average energy per particle is 3/5 of the Fermi energy.

2.3 Sommerfeld expansion

Let us now look at finite temperatures. For high temperature, the calculations are the same as those in boson gas up to a sign change in the denominator of the distribution. The low temperature expansion, known as the Sommerfeld expansion, is more interesting.

However, before doing the math (which is very messy), let us first think about the intuition behind. Intuitively, we have to excite the particles near the Fermi surface first. With low temperature T, only particles with energy around k_BT lower than the Fermi energy are excited. There are $\rho(E_F)k_BT$ such particles, where $\rho(E_F)$ is the density of state at the Fermi energy. If the excitation energy is also around k_BT, the total energy gained is

$$\Delta E \sim \rho(E_F)(k_BT)^2\,. \tag{35}$$

Hence, the heat capacity

$$C_V \sim \rho(E_F)k_B^2T \sim \frac{Nk_B^2T}{E_F}\,, \tag{36}$$

is linear in T. This is the most important take-home message.

Now we rigorously derive C_V with the Sommerfeld expansion. We first write down the condition for z:

$$N = \frac{gV}{4\pi^2}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty \frac{x^{1/2}\,dx}{z^{-1}e^x+1}\,. \tag{37}$$

Similarly, one can derive that the total energy is

$$E = \frac{gV}{4\pi^2\beta}\left(\frac{2m}{\hbar^2\beta}\right)^{3/2}\int_0^\infty \frac{x^{3/2}\,dx}{z^{-1}e^x+1}\,. \tag{38}$$

It remains to see how to expand the integral

$$g_n(z) = \int_0^\infty \frac{x^n\, dx}{z^{-1}e^x + 1}\,, \tag{39}$$

for large z, as $z \to \infty$ at $T = 0$. First, it is useful to change the variable

$$g_n(z) = \int_{-\log z}^\infty \frac{(x + \log z)^n\, dx}{e^x + 1}\,. \tag{40}$$

Using integration by parts

$$g_n(z) = \left[\frac{(x + \log z)^{n+1}}{(n+1)(e^x + 1)}\right]_{-\log z}^\infty - \int_{-\log z}^\infty \frac{(x + \log z)^{n+1}\, dx}{n+1} \frac{d}{dx}\frac{1}{e^x + 1} \tag{41}$$

$$= \int_{-\log z}^\infty \frac{(x + \log z)^{n+1}\, dx}{n+1} \frac{e^x}{(e^x + 1)^2}\,. \tag{42}$$

Since the integrand decays exponentially for large negative x, we can extend the integration range to $(-\infty, \infty)$. We can also Taylor expand the binomial $(x + \log z)^{n+1}$:

$$g_n(z) \simeq \int_{-\infty}^\infty dx \left(\frac{(\log z)^{n+1}}{n+1} + x(\log z)^n + \frac{n}{2}x^2(\log z)^{n-1} + \cdots\right)\frac{e^x}{(e^x + 1)^2}\,. \tag{43}$$

The first few terms in the expansion dominate as z is large. Note that the x term vanishes under the integral as $e^x/(e^x + 1)^2$ is even. Therefore,

$$g_n(z) \simeq \frac{(\log z)^{n+1}}{n+1}\left(1 + \frac{\pi^2}{6}\frac{(n+1)n}{(\log z)^2} + \cdots\right)\,. \tag{44}$$

We can use the above result to find the heat capacity of free fermion gas at low temperature. First, since N does not change with temperature, we require

$$\frac{g_{1/2}(z)}{\beta^{3/2}} = \text{const.} \quad \Rightarrow \quad \mu^{3/2}\left(1 + \frac{\pi^2}{8}\left(\frac{k_BT}{\mu}\right)^2\right) = \text{const.}. \tag{45}$$

Since we know that $\mu(T = 0) = E_F$, we get

$$\mu \simeq E_F\left(1 - \frac{\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2\right)\,. \tag{46}$$

Then, we can calculate the total energy in the following way:

$$\frac{E}{N} = \frac{1}{\beta}\frac{g_{3/2}(z)}{g_{1/2}(z)} \tag{47}$$

$$\simeq \frac{3}{5}\mu\left(1 + \frac{\pi^2}{2}\left(\frac{k_BT}{\mu}\right)^2\right) \tag{48}$$

$$= \frac{3}{5}E_F\left(1 + \frac{5\pi^2}{12}\left(\frac{k_BT}{E_F}\right)^2\right)\,. \tag{49}$$

Finally, the heat capacity is

$$C_V = \frac{\partial E}{\partial T} = \frac{\pi^2}{2}\frac{Nk_B^2T}{E_F}\,, \tag{50}$$

which indeed agrees with our intuition in the above, but now the coefficient is exact.

3 Superfluid helium-4

Isotopes of helium give interesting examples of both boson and fermion gases, as remarkably, helium does not freeze into solid even at zero temperature under room pressure. The lighter isotope helium-3 has two protons and one neutrons, so it is a fermion and there is no condensation at low temperature. As a result, there is no superfluid phase transition for helium-3 (except when the atoms pair up to form effective bosons at even lower temperature, similar to superconductivity).

The heavier isotope helium-4 has two protons and two neutrons, so it is a boson and there should be Bose-Einstein condensation (BEC) at low temperature. Correspondingly, there is a superfluid phase transition around $T \simeq 2.17$ K, which is indeed close to the BEC transition temperature (3.14 K). Below the transition temperature, some of the helium atoms enter the superfluid phase, again in a way similar to BEC.

However, there are strong interactions between the helium atoms, so the superfluid phase is not fully described by the BEC for free bosons. In particular, it is the interactions that lead to zero viscosity in the condensate. A clear difference between the free and interacting systems is in the behavior of heat capacity. For the superfluid, the heat capacity diverges at the transition temperature, and vanishes as T^3 (like phonons) when $T \to 0$. In contrast, the heat capacity of the free BEC is finite at the transition temperature, and vanishes as $T^{3/2}$ when $T \to 0$.

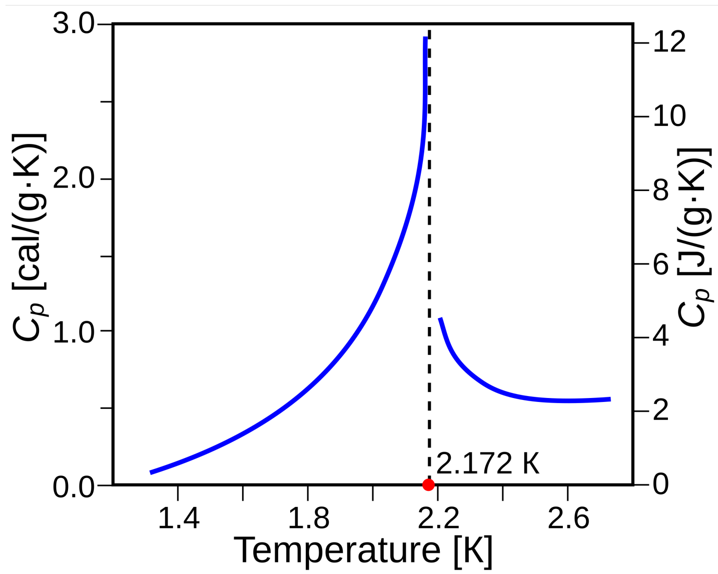

Figure 1:The heat capacity of helium-4 as a function of temperature.